Linear stability analysis of piezoelectric controlled beams subjected to nonconservative loads

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SUMMARY. Discrete model of piezoelectric controlled beams, subjected to follower forces, are studied in this paper. The failure of a similar controller in improving stability properties of the beam is also discussed, this being in contrast with the common practice in the use of piezoelectric devices for which similarity is requested in order to maximize the energy exchange between structure and controller. Moreover, three novel linear gyroscopic controllers are proposed and, by making use of eigenvalues sensitivity analysis, their effectiveness is discussed.

1 Introduction

Vibrations propagating into a structural component can be the source of many undesirables effects. Moreover, the presence of nonconservative loads can affect the stability of the system [1], leading to uncontrolled vibrations that can affect the reliability and durability of mechanical structures, increasing fatigue load, and consequently the risk of damaging the structure. Classical techniques adopted for reducing vibrations in structures are essentially based on the use of passive dissipative materials, e.g. viscoelastic layers bonded on the structure, or on added mass. While these techniques have the advantage of being easy to optimize, they have as main drawback the problem of making the structure heavier, especially when trying to efficiently reduce low frequency vibrations. This is usually in counter-position with the low-weight requirements, especially for aeronautic and aerospace applications. In order to overcome these difficulties, the research on piezoelectric transducers increased over the last twenty years. The advantages of using active materials with respect to traditional solutions are reduced mass and weight, high performances and adaptability. It is important to underline that piezoelectric transducers can be regarded, at the same time, as sensors and actuators.

Within the framework of vibration control, one can distinguish between two main strategies: active and passive (see e.g. [2, 3] and [4] for basic concepts of active and passive control, respectively). Both types of controllers have been object of several studies, and each of them has its advantages and drawbacks. In particular, one has to underline that active controllers are very sensitive to parameters mismatch, and this has a negative influence on the stability of the system. On the other hand for passive controllers, the controller cannot give energy to the mechanical system and, as a consequence, a mismatch in parameters cannot affect the stability of the system, if we are considering a conservative system. This is evident when considering passive distributed controllers which are the electrical analog of the mechanical structure (i.e. they are governed by the same PDEs), as those presented in [5, 6, 7]. The concept on which these systems are based exploits a network of uniformly distributed piezoelectric transducers to transform strain energy into capacitive energy: this last will be subsequently dissipated using resistors, the optimal value of which is determined by a pole placement criterion. The application of the same concept to a discrete system is straightforward, as continuum...
field equations are actually obtained starting from a discrete network of actuators after a suitable homogenization procedure [8].

While the effect of active controllers on stability of structures subjected to nonconservative loads has been investigated in the literature, see e. g. [9, 10], the behavior induced by passive controllers has not yet been deeply explored. We will show in this paper that symmetric (or similar) passive controllers, which are the best choice for maximizing the energy exchange between mechanical and electrical parts, have a negative effect on stability in presence of nonconservative loads.

The paper is structured as follows. In Section 2 a discrete model of a piezoelectric controlled beam is derived. In Section 3 the effects on stability of a similar controller are presented. In Section 4 the key idea in searching for a control strategy is briefly discussed. In Section 5 three novel linear gyroscopic controllers are proposed. In Section 6 some numerical results confirming the correct choice of the controllers are presented, while in Section 7 some conclusion are drawn.

2 The discrete model of a piezoelectric controlled structure

The equations of motion for a discrete piezoelectric controlled structure, in presence of nonconservative follower forces, can be derived via the Extended Hamilton’s principle:

\[
\delta H = \delta H_m + \delta H_e + \delta H_{em} = 0
\]

in which, the subscripts “m”, “e” and “em” refer to mechanical, electrical and electro-mechanical quantities, respectively:

\[
\begin{align*}
\delta H_m & := \int_{t_1}^{t_2} \left( \delta \dot{X}^T M_m \dot{X} - \delta X^T K_m X - \delta X^T C_m \dot{X} - \mu_m \delta X^T H_m X \right) dt \\
\delta H_e & := \int_{t_1}^{t_2} \left( \delta \dot{Y}^T M_e \dot{Y} - \delta Y^T K_e Y - \delta Y^T C_e \dot{Y} - \mu_e \delta Y^T H_e Y \right) dt \\
\delta H_{em} & := \int_{t_1}^{t_2} \left( -\gamma \delta \dot{Y}^T G \dot{X} + \delta X^T G Y \right) dt
\end{align*}
\]

where \(X\) and \(Y\) are \(n\)-dimensional vectors of Lagrangian mechanical and electrical coordinates, respectively; \(M_m, K_m, C_m, H_m\) are the \((n \times n)\) mass, stiffness, damping and circulatory matrices for mechanical \((i = m)\) and electrical \((i = e)\) systems, respectively; \(G\) is the gyroscopic matrix, \(\mu_i\) is the nonconservative load parameter for mechanical \((i = m)\) and electrical \((i = e)\) systems and \(\gamma\) is the coupling electro-mechanical parameter. While \(M_e, K_e, C_e, H_e\) are symmetric, the \(H_i\) matrices are not symmetric.

By integrating by parts, the equations of motion follow:

\[
\begin{cases}
M_m \ddot{X} + C_m \dot{X} + (K_m + \mu_m H_m) X - \gamma G \dot{Y} = 0 \\
K_e Y + C_e \dot{Y} + (K_e + \mu_e H_e) Y + \gamma G X = 0
\end{cases}
\]

3 Failure of a similar piezoelectric controller

By taking: \(M_m := M, M_e := \nu_e M, K_m := K, K_e := \kappa_e K, C_m := C, C_e := \xi_e C, H_m \equiv H := H, \mu_m = \mu_e := \mu\), Eqs (3) read:

\[
\begin{cases}
M \ddot{X} + C \dot{X} + (K + \mu H) X - \gamma G \dot{Y} = 0 \\
\nu_e M \ddot{Y} + \xi_e C \dot{Y} + (\kappa_e K + \mu H) Y + \gamma G X = 0
\end{cases}
\]

When the mechanical system is uncoupled from the electrical one, i.e. \(\gamma = 0\), it admits, at a critical load value \(\mu_0\), two purely imaginary eigenvalues \(\lambda_0 = \pm i \omega_0\). When a similar electric
controller is introduced, the mechanical eigenvalue $\lambda_0$ (with fixed $\mu_0$), is modified. We want to study the effects of a similar electric controller on the eigenvalue $\lambda_0$.

First, let us introduce a set of electrical parameter such that: $\nu_\epsilon = 1 + \nu_\epsilon, \xi_\epsilon = 1 + \xi_\epsilon, \kappa_\epsilon = 1 + \kappa_\epsilon$, in which $\eta_i (i = \nu, \xi, \kappa)$ are deviations (imperfections) from perfect similarity. By letting $X = xe^M$, $Y = ye^M$, an eigenvalue problem follows from Eqs (4):

\[
\begin{cases}
(\lambda^2 e^M + \lambda e^M + K + \mu_0 H) x - \lambda \gamma G y = 0 \\
(\lambda^2 (1 + \eta_\nu) e^M + \lambda (1 + \eta_\xi) C + (1 + \eta_\kappa) K + \mu_0 H) x + \lambda \gamma G y = 0
\end{cases}
\]

Imperfections and coupling parameters are rescaled as $\lambda I$, where $\eta \ll 1$ is a perturbation parameter; eigenpairs are expanded in Maclaurin series: $\lambda = \lambda_0 + \varepsilon \lambda_1 + O (\varepsilon^2)$, $x = x_0 + \varepsilon x_1 + O (\varepsilon^2), y = y_0 + \varepsilon y_1 + O (\varepsilon^2)$. The following perturbation equations are obtained:

\[
\begin{align*}
\varepsilon^0 : & \quad \begin{cases}
(\lambda_0^2 e^M + \lambda_0 e^M + K + \mu_0 H) x_0 = 0 \\
(\lambda_0^2 e^M + \lambda_0 e^M + C + \mu_0 H) y_0 = 0
\end{cases} \\
\varepsilon^1 : & \quad \begin{cases}
(\lambda_0^2 e^M + \lambda_0 e^M + K + \mu_0 H) x_1 = -\lambda_1 (2\lambda_0 e^M + C) x_0 + \gamma \lambda_0 G y_0 \\
(\lambda_0^2 e^M + \lambda_0 e^M + C + \mu_0 H) y_1 = -\lambda_1 (2\lambda_0 e^M + C) y_0 - \gamma \lambda_0 G x_0 + \\
& \quad - (\lambda_0^2 \eta_\nu e^M + \lambda_0 \eta_\kappa C + \eta_\kappa K) y_0
\end{cases}
\end{align*}
\]

Since $\lambda_0$ is a semi-simple eigenvalue for the controlled system, the solution of the $\varepsilon^0$-order problem is:

\[
\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = a_1 \begin{pmatrix} u \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ u \end{pmatrix}
\]

in which $a_1$ and $a_2$ are unknowns amplitude, $u$ is a right mechanical (or electrical) eigenvector, satisfying $(\lambda_0^2 e^M + \lambda_0 e^M + K + \mu_0 H) u = 0$. The $\varepsilon^0$-order problem admits also two left eigenvectors, namely $(v_0, 0)^T$ and $(0, v)^T$, in which $v$ is a left mechanical (or electrical) eigenvector, satisfying $(\lambda_0^2 e^M + \lambda_0 e^M + C + \mu_0 H) v = 0$ and normalized via $v^H u = 1$. Solvability condition at $\varepsilon$-order, leads to:

\[
(L - \lambda_1 I) a = 0
\]

where $c := v^H Cu$, $b := v^H Hu$, $m := v^H Mu$, $g := v^H Gu$, $a := \{a_1, a_2\}^T$, and:

\[
L = \frac{1}{2\lambda_0 m + c} \begin{pmatrix} 0 & \gamma \lambda_0 g \\
-\gamma \lambda_0 g & (\lambda_0^2 \eta_\nu m + \lambda_0 \eta_\kappa c + \eta_\kappa k)
\end{pmatrix}
\]

This is an eigenvalue problem for $\lambda_1$, which is determined by the characteristic equation:

\[
\lambda_1^2 + Z_1 \lambda_1 + Z_2 = 0
\]

where $Z_1 := -\text{Tr}(L)$ and $Z_2 := \det(L)$ are the linear and quadratic invariants of $L$, respectively. Eq (10) admits two (generally) distinct roots $\lambda_1^\pm$, depending on the electro-mechanical coupling and imperfections, $\lambda_1^\pm = \lambda_1^\pm (\gamma, \eta_\nu, \eta_\xi, \eta_\kappa)$. Asymptotic stability of the system requires that $\text{Re}(\lambda_1^+ < 0$; if the electric controller is perfect, i.e. if $\eta_\kappa = \eta_\nu = \eta_\kappa = 0$, the linear invariant $Z_1$ is zero, so that
one of the two roots $\lambda_i^\pm = \pm 2\sqrt{-I_2}$ has positive real part. This means that a similar controller has always a detrimental effect on stability, since it splits the eigenvalue $\lambda_0$ giving rise to an unstable eigenvalue. If some imperfections (of an appropriate sign) $\eta_i$ are introduced, then, if $I_1 > 0$, it is possible that $Re(\lambda_i^+) < 0$, this entailing re-stabilization. However, it could be proved that:

1. not all imperfections $\eta_i$, when acting alone, have a beneficial effect on stability;
2. in the space of the imperfection parameters can exist (generally small) regions where imperfections are stabilizing.

Due to these reasons, the contribution given by the imperfections in stabilizing the system is modest and, in order to have a controller which have a significant incremental effect on stability, it is necessary to give up the idea of similarity.

4 Looking for a control strategy

We want to explore the set of unsimilar controllers for which: 1) the destabilizing effect due to the follower electrical force is eliminated, i.e. $\mu = 0$ and, 2) the load value of the mechanical follower force is set to the critical value $\mu_0$. With these assumptions, equations of motion (3) become:

\[
\begin{align*}
\dot{M}_m \ddot{X} + C_m \dot{X} + (K_m + \mu_0 H_m) X - \gamma G \dot{Y} &= 0 \\
M_e \ddot{Y} + C_e Y + K_e Y + \gamma G \ddot{X} &= 0
\end{align*}
\]  

(11)

In looking for a control strategy, we think to perform the following steps: 1) the mechanical structure oscillates with a certain amplitude and frequency $\omega_0$, while the electrical controller is turned off; 2) the controller is turned on, and its motion is forced by the large response of the mechanical system; 3) the mechanical structure is modified by the response of the electrical controller, via the gyroscopic coupling. The task is in maximizing the amplitude of the response of the electrical oscillator, in order it can dissipate energy. Accordingly, we perform a straightforward expansion of the eigenvalues $\lambda_0 = \pm i\omega_0$, and we look for singularities which make the electrical response infinite.

By letting $X = xe^{\lambda t}$, $Y = ye^{\lambda t}$, we obtain an eigenvalue problem; then, we order the coupling as $\gamma \rightarrow \varepsilon \gamma$ and expand the eigenpairs as $\lambda = \lambda_0 + \varepsilon^2 \lambda_2 + O(\varepsilon^3)$, $x = x_0 + \varepsilon^2 x_2 + O(\varepsilon^3)$, $y = \varepsilon y_1 + \varepsilon^3 y_3 + O(\varepsilon^4)$; the perturbation equations follow:

\[
\begin{align*}
\varepsilon^0 : \left( \lambda_0^2 M_m + \lambda_0 C_m + K_m + \mu_0 H_m \right) x_0 &= 0 \\
\varepsilon^1 : \left( \lambda_0^2 M_e + \lambda_0 C_e + K_e \right) y_1 &= -\gamma \lambda_0 G x_0 \\
\varepsilon^2 : \left( \lambda_0^2 M_m + \lambda_0 C_m + K_m + \mu_0 H_m \right) x_2 &= \gamma \lambda_0 G y_1 - \lambda_2 (2\lambda_0 M_m + C_m) x_0
\end{align*}
\]  

(12)

Since $\lambda_0$ is a simple eigenvalue for the mechanical system, the $\varepsilon^0$-order solution is $x_0 = u_m$, where $u_m$ is the right mechanical eigenvector, satisfying $\left( \lambda_0^2 M_m + \lambda_0 C_m + K_m + \mu_0 H_m \right) u_m = 0$; moreover, a left mechanical eigenvector $v_m$ exists, stemming from $\left( \lambda_0^2 M_m^T + \lambda_0 C_m^T + K_m^T + \mu_0 H_m^T \right) v_m = 0$, and normalized via $v_m^H u_m = 1$. The $\varepsilon$-order problem supplies:

\[
y_1 = -\gamma \lambda_0 S^{-1}_c G u_m
\]  

(13)

in which $S_c := \left( \lambda_0^2 M_e + \lambda_0 C_e + K_e \right)$ is a not singular matrix if $\lambda_0$ is not an eigenvalue of the controller system. Solvability condition at the $\varepsilon^2$-order furnishes:

\[
\lambda_2 = -\frac{\gamma^2 \lambda_0^2 g_m}{2\lambda_0 m_m + c_m}
\]  

(14)
where \( c_m := v_m^H C_m u_m, \) \( m_m := v_m^H M_m u_m, \) \( g_m := v_m^H G \mathbf{S}_e^{-1} G u_m. \) For stability we need \( \Re(\lambda_2) < 0 \) and, in order to maximize the beneficial effect \( |\Re(\lambda_2)| \to \infty. \) Since \( \gamma \) is a quantity of order \( \varepsilon, \) while \( \lambda_0, m_m \) and \( c_m \) are of order 1, it needs that \( |\Re(g)| \to \infty \) or, equivalently, \( |y_1| \to \infty, \) i.e. the electrical response is large.

To maximize the electrical response, we come back to the \( \varepsilon \)-order problem, assume a Rayleigh-type damping, for which \( C_e := (\alpha_e K_e + \beta_e M_e), \) where \( \alpha_e \) and \( \beta_e \) are damping coefficients, and rewrite the equations in the basis \( U_e \) of the right eigenvectors; we obtain:

\[
\begin{bmatrix}
-\omega_e^2 \hat{M}_e + i \omega_0 (\alpha_e \hat{K}_e + \beta_e \hat{M}_e) + \hat{K}_e \\
\end{bmatrix} \hat{y}_1 = \hat{f}
\]

where \( \hat{y}_1 := U_e^T y_1 \) is the modal amplitude, \( \hat{M}_e := U_e^T M_e U_e = \text{diag}(m_1^e, \ldots, m_n^e), \) \( \hat{K}_e := U_e^T K_e U_e = \text{diag}(k_1^e, \ldots, k_n^e) \) are the mass and stiffness modal matrices, respectively, and \( \hat{f} := -\gamma \lambda_0 U_e^T G u_m. \) The \( j \)-th equation of system (15) furnishes:

\[
|\hat{y}_j| = \frac{|\hat{f}_j|}{\sqrt{(k_j^e - \omega_0^2 m_j^e)^2 + (\xi_j^e)^2}}
\]

(16)

where \( \xi_j^e := \alpha_e k_j^e + \beta_e m_j^e \) is the \( j \)-component of the modal Rayleigh-type damping.

From Eq (16) it follows that, in order for \( |\hat{y}_j| \to \infty, \) and by assuming \( \xi_j^e \) sufficiently small, different strategies can be designed, by taking:

1. **Non-Singular (resonant) Controller (NSC):**
   \( k_j^e = O(1), \) \( m_j^e = O(1) \) and \( \omega_0^2 = k_j^e / m_j^e + O(\varepsilon); \)
   moreover, \( \xi_j^e = O(\varepsilon); \)

2. **Singular Non-Resonant Controller (SNRC):**
   \( k_j^e = O(\varepsilon), \) \( m_j^e = O(\varepsilon) \) and \( k_j^e / m_j^e \neq \omega_0^2, \forall j; \)
   moreover, \( \xi_j^e = O(\varepsilon); \)

3. **Singular Resonant Controller (SRC):**
   \( k_j^e = O(\varepsilon), \) \( m_j^e = O(\varepsilon) \) and \( k_j^e / m_j^e = \omega_0^2 \) for a given \( j; \)
   moreover, \( \xi_j^e = O(\varepsilon^{3/2}); \)

All the controllers resemble passive devices well-known in literature, namely: the NSC and the SRC are similar to the Tuned Mass Damper [11], with large or small mass and stiffness, respectively; the SNRC is similar to the Nonlinear Energy Sink [12]. In all cases, however, differently from the classical equipments, coupling is **linear** and **gyroscopic**. For all these systems, the straightforward expansion used here fail, and **ad hoc** eigenvalue expansions must be adopted, as discussed ahead.

5 **Gyroscopic controlling system**

We consider Eqs (3), by taking \( M_m := M, \) \( M_e := \nu_e M, \) \( K_m := K, \) \( K_e := \kappa_e K, \) \( C_m := C, \) \( C_e := \xi_e C, \) \( H_m := H, \) \( \mu_m := \mu, \) \( \mu_e := 0, \) i.e.:

\[
\begin{aligned}
M \ddot{X} + C \dot{X} + (K + \mu H) X - \gamma G \dot{Y} &= 0 \\
\nu_e M \ddot{Y} + \xi_e C \dot{Y} + \kappa_e K Y + \gamma G X &= 0
\end{aligned}
\]

(17)

By letting \( X = x e^{\lambda t}, \) \( Y = y e^{\lambda t}, \) the eigenvalue problem follows:

\[
\begin{aligned}
(\lambda^2 M + \lambda C + K + \mu H) x - \lambda \gamma G y &= 0 \\
(\lambda^2 \nu_e M + \lambda \xi_e C + \kappa_e K) y + \lambda \gamma G x &= 0
\end{aligned}
\]

(18)
We are aimed to find the eigenvalue $\lambda$ as a perturbation of $\lambda_0 = i\omega_0$, for the three controllers described above. To this end, a sensitivity analysis must be carried out in different cases, in which: 1) NSC, $\lambda_0$ is a semi-simple eigenvalue, 2) SNRC, $\lambda_0$ is a simple eigenvalue, 3) SRC, $\lambda_0$ is a non-semisimple (defective) eigenvalue [13].

5.1 The leading order solution

In order to stress the differences among the three problems stated above, we discuss them all together, by limiting ourselves, for the moment, to the (generating) lower-order solutions.

Non-Singular Controller (NSC). We take $\nu_e = 1$ and perform the rescalings $\xi_e \rightarrow \varepsilon \xi_e$, $\gamma \rightarrow \varepsilon \gamma$. Moreover, we adjust $\kappa_e$ to tune an electric eigenvalue to the resonant value. At the leading-order the eigenvalue problem reads:

$$
\begin{bmatrix}
S_m(\lambda_0) & 0 \\
0 & S_e(\lambda_0)
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

where:

$$
S_m(\lambda_0) := \lambda_0^2 M + \lambda_0 C + K + \mu_0 H
$$

$$
S_e(\lambda_0) := \lambda_0^2 M + \kappa_e K
$$

are algebraic mechanical and electrical operators, respectively. The semi-simple eigenvalue $\lambda_0$ admits two distinct right eigenvectors, so that:

$$
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
= a_m \begin{bmatrix} u_m \\ 0 \end{bmatrix} + a_e \begin{bmatrix} 0 \\ u_e \end{bmatrix}
$$

where the amplitudes $a_m$ and $a_e$ are arbitrary and $u_m; S_m u_m = 0$ is a right mechanical eigenvector and $u_e; S_e u_e = 0$ is a right electrical eigenvector, generally different each other. Moreover, problem (19) admits two left eigenvectors, namely $\left\{ v_m, 0 \right\}^T$ and $\left\{ 0, v_e \right\}^T$, where $v_m; S_m^H v_m = 1$, $v_e; S_e^H v_e = 1$ enforced. Since the electric oscillator is Hamiltonian, $v_e \equiv u_e$. It is concluded that, the mechanical and electric responses are of the same order.

Singular Non-Resonant (SNRC) and Resonant (SRC) Controllers. The parameters $\nu_e, \kappa_e, \gamma$ are rescaled via $\nu_e \rightarrow \varepsilon \nu_e, \kappa_e \rightarrow \varepsilon \kappa_e, \gamma \rightarrow \varepsilon \gamma$; moreover, in the SNRC $\xi_e \rightarrow \varepsilon \xi_e$ and in the SRC $\xi_e \rightarrow \varepsilon^{3/2} \xi_e$ are taken. To make this latter resonant, the ratio $\kappa_e / \nu_e$ is fixed. Since the second equation has evanescent coefficients, we divide it by $\nu_e$, and take the leading terms, so obtaining:

$$
\begin{bmatrix}
S_m(\lambda_0) & 0 \\
\frac{2\lambda_0}{\nu_e} G & S_e(\lambda_0)
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

where $S_m$ is defined by Eq (20) and, moreover:

$$
S_e(\lambda_0) := \begin{cases}
\lambda_0^2 M + \lambda_0 \frac{\kappa_e}{\nu_e} C + \frac{\kappa_e}{\nu_e} K & \text{for SNRC} \\
\lambda_0^2 M + \frac{\kappa_e}{\nu_e} K & \text{for SRC}
\end{cases}
$$

In the SNRC, $\lambda_0$ is a simple eigenvalue, since $\det [S_m(\lambda_0)] = 0$ and $\det [S_e(\lambda_0)] \neq 0$; therefore:

$$
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2\lambda_0 C + \frac{\kappa_e}{\nu_e} K} \frac{u_m}{\nu_e} S_e^{-1} G u_m \\
\frac{1}{2\lambda_0 C + \frac{\kappa_e}{\nu_e} K} \frac{u_m}{\nu_e} S_e^{-1} G u_m
\end{bmatrix}
$$
where \( u_m : S_m u_m = 0 \). Although the mechanical and electric responses are of the same order, the electric oscillator behaves as passive, driven by the mechanical one.

In the SRC, \( \lambda_0 \) is a double eigenvalue, since \( \det [S_m(\lambda_0)] = \det [S_e(\lambda_0)] = 0 \); however, just one proper right eigenvector, \( \{0, u_e\}^T \) exists, with \( u_e : S_e u_e = 0 \), so that \( \lambda_0 \) is defective. Similarly, there is just one proper left eigenvector, \( \{v_m, 0\}^T \). Therefore, the leading-order solution is:

\[
\begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  u_e
\end{pmatrix}
\]

which shows that the electric response is larger than the mechanical response!

In the following subsections, the perturbation expansions are carried out at higher-orders.

5.2 The Non-Singular Controller

The following Mc Laurin series expansions are adopted:

\[
\begin{align*}
\mu &= \mu_0 + \varepsilon \mu_1 + O(\varepsilon^2) \\
\lambda &= \lambda_0 + \varepsilon \lambda_1 + O(\varepsilon^2) \\
x &= x_0 + \varepsilon x_1 + O(\varepsilon^2) \\
y &= y_0 + \varepsilon y_1 + O(\varepsilon^2)
\end{align*}
\]

By substituting them into the eigenvalue problem (18) and by making use of Eq (20), Eq (19) is obtained, together with:

\[
\varepsilon^1 : \begin{bmatrix} S_m(\lambda_0) & 0 \\ 0 & S_e(\lambda_0) \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \gamma \lambda_0 G y_0 - \lambda_1 (2\lambda_0 M + C) x_0 - \mu_1 H x_0 \\ -\gamma \lambda_0 G x_0 - \lambda_0 (2\lambda_1 M + \xi_e C) y_0 \end{pmatrix}
\]

Once the solution (21) is substituted into this latter and solvability is required, an algebraic eigenvalue problem follows:

\[
L = \begin{bmatrix}
\frac{\mu_1 h_{mm}}{2\lambda_0 m_{mm} + \tau_{mm}} & \frac{\gamma \lambda_0 h_{me}}{2\lambda_0 m_{mm} + \tau_{mm}} \\
\frac{-\mu_1 h_{me}}{2\lambda_0 m_{ee} + \tau_{ee}} & \frac{-\gamma \lambda_0 h_{me}}{2\lambda_0 m_{ee} + \tau_{ee}}
\end{bmatrix}
\]

where the following definitions have been introduced:

\[
\begin{align*}
m_{mm} &:= v_m^H M u_m, \quad m_{ee} := v_e^H M u_e, \quad c_{mm} := v_m^H C u_m, \quad c_{ee} := u_e^H C u_e \\
h_{mm} &:= v_m^H H u_m, \quad g_{me} := v_m^H G u_e, \quad g_{em} := u_e^H G u_m
\end{align*}
\]

The characteristic equation of problem is still of the form (10), where \( I_1 = I_1(\mu_1, \xi_e) \neq 0, \forall (\mu_1, \xi_e) \neq (0, 0) \) and \( I_2 = I_2(\mu_1, \xi_e, \gamma) \); it admits generally distinct roots \( \lambda_1^{\pm} = \lambda_1^{\pm}(\mu_1, \xi_e, \gamma) \).

The NSC is asymptotically stable if \( \text{Re} \left( \lambda_1^{\pm} \right) < 0 \), i.e if (Bilharz Theorem):

\[
\begin{align*}
\text{Re} (I_1) &> 0 \\
\text{Im} (I_2)^2 - \text{Re} (I_2) \text{Im} (I_1) \text{Im} (I_2) - \text{Re} (I_1)^2 &< 0
\end{align*}
\]

Eqs (30) define the stability domain of the system; the stability boundary \( F(\mu_1, \xi_e, \gamma) = 0 \) is a surface in the 3-D parameters space.
5.3 The Singular Non-Resonant Controller
The same Eqs (26) are used, leading to Eq (22) and:

\[
\varepsilon^1 : \begin{bmatrix} S_m(\lambda_0) & 0 \\ \frac{\partial}{\partial \varepsilon} G & S_c(\lambda_0) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \gamma \lambda_0 G y_0 - \lambda_1 (2\lambda_0 M + C) x_0 - \mu_1 H x_0 \\ -\frac{\gamma}{\varepsilon} \lambda_0 G x_1 + \lambda_1 (2\lambda_0 M + \frac{\varepsilon}{\varepsilon} C) y_0 \end{bmatrix}
\]

By accounting for the leading solution (24), solvability condition of the $\varepsilon$-order problem furnishes:

\[
\lambda_1 = -\frac{\mu_1 \varepsilon h_{mm} + \gamma^2 \lambda_{0}^2 g_{mm} (\nu_c, \xi_c, \kappa_c)}{\nu_c (2\lambda_0 m_{mm} + c_{mm})}
\]

where $h_{mm}$, $c_{mm}$, and $m_{mm}$ are defined in Eq (29), while $g_{mm} (\nu_c, \xi_c, \kappa_c) = v_m^H G \varepsilon^{-1} G u_m$.

The SNRC is asymptotically stable if $\text{Re} (\lambda_1) < 0$; even if the critical load is incremented, a re-stabilizing effect is always possible. The critical condition for the system is $\text{Re} (\lambda_1) = 0$, which determines the hypersurface $\mathcal{F} (\mu, \nu_c, \xi_c, \kappa_c, \gamma) = 0$ in the 5-D space of parameters.

5.4 The Singular Resonant Controller
For the SRC the following Newton-Puiseux series expansions hold:

\[
\begin{align*}
\mu &= \mu_0 + \varepsilon^{1/2} \mu_{1/2} + \varepsilon \mu_1 + O (\varepsilon^{3/2}) \\
\lambda &= \lambda_0 + \varepsilon^{1/2} \lambda_{1/2} + \varepsilon \lambda_1 + O (\varepsilon^{3/2}) \\
x &= x_0 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + O (\varepsilon^{3/2}) \\
y &= y_0 + \varepsilon^{1/2} y_{1/2} + \varepsilon y_1 + O (\varepsilon^{3/2})
\end{align*}
\]

From these, Eqs (22) follow, together with:

\[
\varepsilon^{1/2} : \begin{bmatrix} S_m(\lambda_0) & 0 \\ \frac{\partial}{\partial \varepsilon} G & S_c(\lambda_0) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda_{1/2} (2\lambda_0 M + C) x_0 + \mu_{1/2} H x_0 \\ \lambda_{1/2} (2\lambda_0 M y_0 + \frac{\varepsilon}{\varepsilon} G x_0) + \frac{\varepsilon}{\varepsilon} \lambda_0 C y_0 \end{bmatrix}
\]

\[
\varepsilon : \begin{bmatrix} S_m(\lambda_0) & 0 \\ \frac{\partial}{\partial \varepsilon} G & S_c(\lambda_0) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 (2\lambda_0 M + C) x_0 + \lambda_{1/2} (2\lambda_0 M + C) x_{1/2} \\ \lambda_1 (2\lambda_0 M y_0 + \frac{\varepsilon}{\varepsilon} G x_0) + \frac{\varepsilon}{\varepsilon} \lambda_0 C y_{1/2} + \lambda_{1/2} C y_0 \\ -\gamma \lambda_0 G y_0 + \mu_1 H x_0 + \mu_{1/2} H x_{1/2} + \lambda_{1/2}^2 M x_0 \\ 2\lambda_0 \lambda_{1/2} M y_{1/2} + \frac{\varepsilon}{\varepsilon} \lambda_{1/2} G x_{1/2} + \lambda_{1/2}^2 M y_0 \end{bmatrix}
\]

Using the leading solution (25), the $\varepsilon^{1/2}$-order problem is found to satisfy the solvability condition, which requires the know term is orthogonal to $\{v_m, 0\}$. It admits the solution:

\[
\begin{bmatrix} x_{1/2} \\ y_{1/2} \end{bmatrix} = -\frac{1}{g_{\text{erm}}^m} (2\lambda_{1/2} \nu_c m_{\text{ee}} + \xi_c c_{\text{ee}}) u_m
\]

where $\hat{y}_{1/2}$ is a particular solution to the singular problem:

\[
S_c(\lambda_0) \hat{y}_{1/2} = \frac{\lambda_0}{g_{\text{erm}}^m} \left( 2\lambda_{1/2} m_{\text{ee}} + \frac{\xi_c}{\nu_c} c_{\text{ee}} \right) G u_m - \lambda_0 \left( 2\lambda_{1/2} M + \frac{\xi_c}{\nu_c} C \right) u_c
\]
rendered unique by a suitable normalization. It should be remarked that, at this order, $\lambda_{1/2}$ is still undetermined. Solvability condition, must instead be invoked at the $\varepsilon$-order problem, which finally determines $\lambda_{1/2}$:

$$\lambda_{1/2}^2 + c_1 \lambda_{1/2} + c_0 = 0 \quad (37)$$

where:

$$c_1 := \frac{\mu_{1/2} h_{mm}}{2\lambda_0 m_{mm} + c_{mm}} + \frac{\xi e c_{ee}}{2\nu_e m_{ee}} \quad (38)$$

$$c_0 := \gamma^2 \lambda_0 g_{mm} g_{ne} + \mu_{1/2} h_{mm} \xi e c_{ee} \frac{2\nu_e m_{ee} (2\lambda_0 m_{mm} + c_{mm})}{2\nu_e m_{ee} (2\lambda_0 m_{mm} + c_{mm})}$$

Equation (37) admits two generally distinct roots $\lambda_{1/2}^{\pm} = \lambda_{1/2}^{\pm} (\mu_{1/2}, \nu_e, \xi_e, \gamma)$, (remember that the ratio $\kappa_e/\nu_e$ is fixed).

The SRC is asymptotically stable if $\text{Re} (\lambda_{1/2}^{\pm}) < 0$. Therefore, since $c_1 \neq 0$, $\forall (\mu_{1/2}, \xi_e) \neq (0, 0)$, even if the critical load is incremented, a re-stabilizing effect is possible. In order for Eq (37) admits roots with real parts less than zero, the Bilharz’s conditions (30) must be satisfied. The SRC’s critical condition, $\text{Re} (\lambda_{1/2}^{\pm}) = 0$, determines a hypersurface $F (\mu_{1/2}, \nu_e, \xi_e, \gamma) = 0$ in the 4-D space of parameters.

6 Numerical Example

Results regarding the effectiveness of the gyroscopic control systems for a 2-D discretized beam, equipped with the three piezoelectric controllers defined above, are displayed in table 1. They show the load increment $\Delta \mu$ (in percentage) with respect to the critical load $\mu_0$, for a fixed set of parameter in each piezoelectric controller. The Hopf conditions for the beam are: $\mu_0 = 4.5$, $\lambda_0 = \pm i$ and the coupling parameter is $\gamma = 0.2$ in all cases. As it can be seen, the Singular Resonant Controller gives the most significant results in terms of the re-stabilizing effect of the beam.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$\nu_e$</th>
<th>$\kappa_e$</th>
<th>$\xi_e$</th>
<th>$\Delta \mu$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSC</td>
<td>1</td>
<td>1.94</td>
<td>0.3</td>
<td>3</td>
</tr>
<tr>
<td>SNRC</td>
<td>0.3</td>
<td>0</td>
<td>0.04</td>
<td>11</td>
</tr>
<tr>
<td>SRC</td>
<td>0.001</td>
<td>0.05</td>
<td>0.01</td>
<td>129</td>
</tr>
</tbody>
</table>

Table 1: Follower load increment in a 2-D controlled beam

7 Conclusions

In this paper the failure of a similar piezoelectric controller, on the improvement in stability properties of a beam, has been discussed. Moreover, three new controller, namely NSC, SNRC and SRC, based on different sensitivities of their eigenvalues, have been introduced and their effectiveness in terms of stability has been showed. However, a more exhaustive parametric analysis and a wide discussion about stability regions and critical surfaces will be presented at the conference time.

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References


