Fractional sensitivities of semi-simple eigenvalues for bifurcation analysis

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SUMMARY. We perform high-order sensitivity analysis of eigenvalues of linear systems depending on parameters. Attention is focused on double not-semi-simimple and semi-simple eigenvalues, undergoing perturbations, either of regular or singular type. The use of integer (Taylor) or fractional (Puiseux) series expansions is discussed, and the analysis carried out on the characteristic polynomial. It is shown that semi-simple eigenvalues can admit fractional sensitivities when the perturbations are singular, conversely to the not-semi-simple case. However, such occurrence only manifests itself when a second-order perturbation analysis is carried out. As a main result, it is found that such over-degenerare case spontaneously emerges in bifurcation analysis, when one looks for the boundaries of the stability domain of circulatory mechanical systems possessing symmetries. A four degree-of-freedom system under a follower force is studied as an illustrative example.

1 Introduction

The dynamics of a linear system depend on its eigenvalues \( \lambda^{(i)}, \ i = 1, 2, \ldots \). Very often, one is not interested in analyzing the behavior of a specific system, but, rather, that of a family of systems, parametrized by one or more control parameters \( \mu \in \mathbb{R}^M \), so that \( \lambda = \lambda(\mu) \) (apex i omitted from now on). Once the eigenvalues \( \lambda_0 := \lambda(\mu_0) \) have been evaluated at a selected point \( \mu_0 \) of the parameter space, one would predict how the eigenvalues vary in a small ball \( \mathcal{R} \) around this point, thus avoiding to repeat the analysis for several values of the increment \( \delta \mu := \mu - \mu_0 \). The problem is of particular importance in bifurcation analysis, where the eigenvalues decide on stability or instability of the equilibrium points.

The task is usually performed by perturbation methods [1]. These require, in the order: (a) to select a family of ‘exploring curves’ \( C \in \mathcal{R} \), of parametric equations \( \mu = \mu(\varepsilon) \), emanating from \( \mu_0 = \mu(0) \), where \( 0 < \varepsilon \ll 1 \) is a perturbation parameter; (b) to assume a (formal) series expansion for the eigenvalue \( \lambda(\mu(\varepsilon)) \), namely:

\[
\lambda = \lambda_0 + \sum_{k=1}^{\infty} \varepsilon^k \lambda_k
\]

(c) to solve in chain several perturbation equations in the unknown coefficients \( \lambda_k \). These latter are called the ‘sensitivities’ of the eigenvalue (by understanding ‘at \( \mu_0 \) along \( C \)’). Since the series expansion only involves integer powers of \( \varepsilon \) (i.e. it is a Taylor series), we will call them integer sensitivities. In degenerate cases, instead, a generalized (or Puiseux) series must be used, involving fractional powers of the perturbation parameter. In the most frequent case of not-derogatory matrix
this series reads:

\[ \lambda = \lambda_0 + \sum_{k=1}^{\infty} \varepsilon^{k/n} \lambda_{k/n} \]  

(2)

When coalescence of the eigenvalues occur and the system is generic, the eigenvalue is defective (also said \textit{not-semi-simple}), this entailing the existence of fractional sensitivities. If, in contrast, suitable geometrical symmetries exist or energy conservation holds, the eigenvalue is not-defective (also said \textit{semi-simple}), so that integer sensitivities are admitted, in spite of the coalescence. However, exceptions are known to exist to this rule, concerning defective eigenvalue [1, 2].

It is now quite natural to ask to ourselves if a converse case, which seems not to have been studied in literature, can occur. Namely: (a) \textit{can a semi-simple eigenvalue admit fractional sensitivities}, as an effect of singular perturbations? And, in the affirmative case, (b) \textit{do singularities determine special system behaviors}, making them worthy of study?

This papers gives affirmative answers to both questions. In Sect 2 a perturbation analysis is carried out on the characteristic polynomial, to display the occurrence of over-degeneracies, and to investigate the mechanism leading to singularity of the expansion. In Sect 3, the theory is specialized to mechanical circulatory systems (i.e. Hamiltonian systems under non-conservative forces), to show the existence of a strict link between singular perturbations and stability boundaries. In Sect 4 a numerical example is worked out. Finally, in Sect 5, some conclusions are drawn.

2 Characteristic polynomial analysis

Let us consider the linear algebraic eigenvalue problem:

\[ (A(\mu) - \lambda I) w = 0 \]  

(3)

in which \( A(\mu) \) is a real square matrix depending on a set of real parameters \( \mu \), \( I \) the identity matrix, \( \lambda \) is a generally complex eigenvalue and \( w \) the associated eigenvector, respectively. Equation (3) calls for solving a characteristic equation of the type:

\[ F(\lambda, \mu) := \det [A(\mu) - \lambda I] = 0 \]  

(4)

We look for an asymptotic expansion of the unknown \( \lambda(\mu) \) around the point \( \mu_0 \) of the parameter space, for which we assume that \( \lambda_0 := \lambda(\mu_0) \) is known. To this end, we explore the neighborhood \( \mathcal{R} \) of \( \mu_0 \) by selected curves \( \mu = \mu(\varepsilon) \), where \( 0 < \varepsilon \ll 1 \) is a perturbation parameter and \( \mu(0) = \mu_0 \). Usually, straight lines \( \mu = \mu_0 + \varepsilon \mu_1 \) are well-suited to the scope, but in some circumstance, to be investigated later, different curves must be used. Along one of these curves, it is \( F(\lambda, \mu(\varepsilon)) \), so that the characteristic equation reads \( F(\lambda, \varepsilon) = 0 \), which admits the pair \( (\lambda_0, 0) \) as a solution, i.e. \( F(\lambda_0, 0) = 0 \).

2.1 Integer sensitivities

We first look for a solution \( \lambda(\varepsilon) \) to Eq. (4) in Taylor series form. By substituting Eq. (1) in Eq. (4), expanding in series, and separately vanishing the terms with the same powers of \( \varepsilon \), the following perturbation equations are derived:

\[
\varepsilon : \quad F^0_\lambda \lambda_1 + F^0_\varepsilon = 0 \\
\varepsilon^2 : \quad 2F^0_\lambda \lambda_2 + F^0_\lambda \lambda_1^2 + 2F^0_\varepsilon \lambda_1 + F^0_\varepsilon = 0 \\
\varepsilon^3 : \quad 6F^0_\lambda \lambda_3 + 6F^0_\lambda \lambda_1 \lambda_2 + F^0_\lambda \lambda_1^3 + 6F^0_\varepsilon \lambda_2 + 3F^0_\lambda \varepsilon \lambda_1^2 + 3F^0_\lambda \lambda_1 \lambda_1 \lambda_2 + F^0_\varepsilon \varepsilon^2 = 0
\]  

(5)
where the subscript denotes differentiation with respect the homonymous variable, and the super-
script 0 evaluation at the starting point \((\lambda_0, 0)\); moreover, \(F_\varepsilon = F^T_\mu \mu_\varepsilon, F_{\varepsilon\varepsilon} = \mu_\varepsilon^T F_{\mu\mu}\mu_\varepsilon + F^T_\mu \mu_{\varepsilon\varepsilon}, \ldots \).

The perturbation equations (5) must be solved in chain. However, several case must be distin-
guished, according to the level of degeneracy. Here, we confine ourselves to simple and double
roots.

1. **Simple root.** If \(\lambda_0\) is a simple root for \(F(\lambda, 0)\), then \(F^0_{\lambda\lambda} \neq 0\). Hence, Eq(5-a) is a linear
   equation for \(\lambda_1\), Eq(5-b) for \(\lambda_2\), and so on; therefore all the coefficients of the series can be
   computed, and singularities do not occur.

2. **Semi-simple double root.** If \(\lambda_0\) is a double root for \(F(\lambda, 0)\), then \(F^0_{\lambda\lambda} = 0\), but \(F^0_{\lambda\lambda\lambda} \neq 0\); moreover, if it is semi-simple, then also \(F^0_{\varepsilon\varepsilon} = 0\). As a matter of fact, if matrix \(A\) (0) contains
   the Jordan block \([ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} ]\), any \(\varepsilon\)–perturbation brings contributions of the type \(\varepsilon(\lambda_0\lambda - \lambda)\)
   and/or \(\varepsilon^2\) to \(F(\lambda, \varepsilon)\), whose \(\varepsilon\)–derivatives all vanish at \((\lambda_0, 0)\). Consequently, Eq(5-a) is
   trivially satisfied, but leaves \(\lambda_1\) undetermined. By going to the \(\varepsilon^2\)–order, a second-degrees
   equation is found for \(\lambda_1\), namely:
   \[
   F^0_{\lambda\lambda\lambda}\lambda_1^2 + 2F^0_{\lambda\varepsilon}\lambda_1 + F^0_{\varepsilon\varepsilon} = 0
   \]  
(6)

   This equation, in not further degenerate cases, supplies two roots \(\lambda_1^\pm\), which denote the splitting
   of the double eigenvalue \(\lambda_0\) in two simple eigenvalues \(\lambda^\pm = \lambda_0 + \varepsilon\lambda_1^\pm + O(\varepsilon^2)\), pro-
   vided \(\lambda_1^\pm \neq \lambda_1^-\). If an enhanced approximation is desired, higher-order equations
   must be solved for each choice of \(\lambda_1^\pm\). We will refer to such solution as the **regular perturbation of a semi-simple eigenvalue**.

3. **Defective double root.** If \(\lambda_0\) is a double root for \(F(\lambda, 0)\) then \(F^0_{\lambda\lambda} = 0, F^0_{\lambda\lambda\lambda} \neq 0\). For
   compatibility of Eq(5-a), \(F^0_{\varepsilon\varepsilon} = 0\) should also be zero, but this circumstance does not occur when
   \(\lambda_0\) is defective and the perturbation is regular (i.e. generic). For example, if matrix \(A\) (0) contains
   the Jordan block \([ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} ]\), an \(\varepsilon\)–perturbation of the \((2, 1)\)-entry makes \(F^0_{\varepsilon\varepsilon} \neq 0\). Therefore, the Taylor series expansion (1) breaks now, and the Puiseux series expansion (2)
   must be used. An exception, however, occurs when the perturbation is singular (i.e. not-
generic). In the previous example, this happens when any entries of \(A\) (0) are \(\varepsilon\)–perturbed, except for the \((2, 1)\)-th. Then, \(F^0_{\varepsilon\varepsilon} = 0\) and the perturbation scheme previously discussed
   involving integer sensitivities, works. We will call such solution the **singular perturbation of a defective eigenvalue**.

According to the classification made in Geometry of a point belonging to a curve of implicit equation
\(F(\lambda, \varepsilon) = 0\) (see e.g. [3] in the context of buckling of elastic structures), we conclude that if \(\lambda_0\) is
a simple root, the point \((\lambda_0, 0)\) is a regular point, i.e. just one curve passes for it (see Fig 1-a); if
\(\lambda_0\) is a defective root generically perturbed, the point \((\lambda, 0)\) is a turning point, i.e. the unique curve
passing for the point has tangent normal to the \(\varepsilon\)–axis (see Fig 1-b); if \(\lambda_0\) is a semi-simple root under
generic perturbations, or it is a defective root under singular perturbation, then the point \((\lambda, 0)\) is a
branch point, at which two curves cross each other (see Fig 1-c).

As well-known in Geometry, however, degenerated branch points could manifest themselves
as cusp points, at which the two intersecting curves share the same tangent (see Fig 1-d). This
occurrence happens, in our perturbation analysis, when \(\lambda_1^+ = \lambda_1^-\), i.e., from Eq. (6), when:

\[
(F^0_{\lambda\varepsilon})^2 - F^0_{\varepsilon\varepsilon} F^0_{\lambda\lambda} = 0
\]  
(7)
entailing $\lambda_1 = -F^0_{\lambda\varepsilon}/F^0_{\lambda\lambda}$. When this degeneracy condition occurs, the double eigenvalue does not split at the $\varepsilon^{-}\text{order}$, so that the singularity of the $\varepsilon^0$-order still persists. The drawback soon manifests itself at the $\varepsilon^3$-order, where the perturbation equation now reads:

$$6 \left( F^0_{\lambda\lambda} \lambda_1 + F^0_{\lambda\varepsilon} \right) \lambda_2 = - \left( 3 F^0_{\lambda\lambda\lambda} \lambda_1 + 3 F^0_{\lambda\lambda\varepsilon} \lambda_1 + 3 F^0_{\lambda\varepsilon\varepsilon} \lambda_1 + F^0_{\varepsilon\varepsilon\varepsilon} \right)$$

(8)

Since the coefficient of $\lambda_2$ is zero, while the 'known term' is in general different from zero, this equation cannot be solved, thus causing the failure of the integer power expansion! Therefore, a fractional series must be employed.

Figure 1: Perturbation of the eigenvalue $\lambda_0$: (a) simple; (b) defective, regularly perturbed; (c) semi-simple, regularly perturbed or defective, singularly perturbed; (d) semi-simple, singularly perturbed.

2.2 Fractional sensitivities

We now try to solve Eq. (4) in Puiseux series, when the Taylor series breaks down, i.e. when $\lambda_0$ is either (a) a regularly perturbed defective eigenvalue (i.e. when $F^0_\lambda = 0$, $F^0_{\lambda\lambda} \neq 0$, $F^0_{\lambda\varepsilon} \neq 0$), or, (b) a singularly perturbed semi-simple eigenvalue (i.e. when $F^0_\lambda = 0$, $F^0_{\lambda\lambda} \neq 0$, $F^0_{\varepsilon} = 0$ and singularity
expressed by Eq. (7)\(^1\). In both cases we use the series (2), with \( n = 2, \) i.e.:

\[
\lambda = \lambda_0 + \varepsilon^{1/2} \lambda_{1/2} + \varepsilon \lambda_1 + \varepsilon^{3/2} \lambda_{3/2} + \varepsilon^2 \lambda_2 + \varepsilon^{5/2} \lambda_5/2 + \varepsilon^3 \lambda_3 + \ldots
\]  

(9)

and obtain the following perturbation equations:

\[
\begin{align*}
\varepsilon & : F^0_{\lambda\lambda} \lambda^2_{1/2} + 2F^0_{\varepsilon} = 0 \\
\varepsilon^{3/2} & : 6F^0_{\lambda\lambda} \lambda_{1/2} \lambda_1 + F^0_{\lambda\lambda\lambda} \lambda^{3}_{1/2} + 6F^0_{\varepsilon \lambda} \lambda_{1/2} = 0 \\
\varepsilon^2 & : F^0_{\lambda\lambda} \left(2\lambda_{1/2} \lambda_{3/2} + \lambda^2_{1}\right) + F^0_{\lambda \lambda \lambda} \lambda^2_{1/2} \lambda_1 + F^0_{\varepsilon \lambda} \lambda_{1/2} + 2F^0_{\varepsilon \varepsilon} \lambda_1 + F^0_{\varepsilon \varepsilon \varepsilon} = 0 \\
\varepsilon^{5/2} & : 2F^0_{\lambda\lambda} \left(\lambda_{1/2} \lambda_2 + \lambda_1 \lambda_{3/2}\right) + F^0_{\lambda \lambda \lambda} \left(\lambda^2_{1/2} \lambda_{3/2} + \lambda_{1/2} \lambda^2_{1}\right) + \\
& \quad \quad + 2F^0_{\varepsilon \lambda} \lambda_{1/2} \lambda_1 + F^0_{\varepsilon \varepsilon \lambda} \lambda_{1/2} + 2F^0_{\varepsilon \varepsilon \varepsilon} \lambda_{3/2} = 0 \\
\varepsilon^3 & : 3F^0_{\lambda\lambda} \left(2\lambda_{1} \lambda_{2} + \lambda^2_3 + 2\lambda_{1/2} \lambda_{5/2}\right) + F^0_{\lambda \lambda \lambda} \left(\lambda^3 + 3\lambda^2_1 \lambda_{2} + 6\lambda_1 \lambda_{1/2} \lambda_{3/2}\right) + \\
& \quad \quad + 3F^0_{\varepsilon \varepsilon \lambda} \left(\lambda^2_1 + 2\lambda_{1/2} \lambda_{3/2}\right) + 6F^0_{\lambda \varepsilon \varepsilon} \lambda_2 + 3F^0_{\varepsilon \varepsilon \varepsilon} \lambda_1 + F^0_{\varepsilon \varepsilon \varepsilon} = 0
\end{align*}
\]  

(10)

where \( F^0_{\lambda} = 0 \) has already been accounted for. We separately consider the two cases.

1. **Regular perturbation of a defective eigenvalue.** Since \( F^0_{\varepsilon} \neq 0, \) Eq. (10-a) supplies two roots:

\[
\lambda^\pm_{1/2} = \pm \sqrt{-2F^0_\varepsilon / F^0_{\lambda\lambda}}
\]  

(11)

which split the coalescent eigenvalues. The higher-order perturbation equations, which are linear in \( \lambda_1, \lambda_{3/2}, \ldots, \) furnish one higher-order sensitivity for each \( \lambda^\pm_{1/2}. \) The two expansions describe the two branches of the curve which merge at the turning point \((\lambda_0, 0)\).

2. **Singular perturbation of a semi-simple eigenvalue.** Since \( F^0_{\varepsilon} = 0, \) from Eq. (10-a) \( \lambda_{1/2} = 0 \) follows. This result identically vanishes Eq. (10-b) and reduces Eq. (10-c) to the Eq. (6) we already found by Taylor expansion. As we observed, in the singular perturbations case (Eq. (7)), just one (double) root \( \lambda_1 = -F^0_{\varepsilon \varepsilon} / F^0_{\lambda\lambda} \) exists. By accounting also for this result, the \( \varepsilon^{5/2} \) -order Eq. (10-d), identically vanishes, while the \( \varepsilon^3 \) -order Eq. (10-e) reduces to:

\[
3F^0_{\lambda \lambda} \lambda^2_{3/2} = - \left(F^0_{\lambda \lambda \lambda} \lambda^3_1 + 3F^0_{\lambda \varepsilon \lambda} \lambda^2_1 + 3F^0_{\varepsilon \varepsilon \lambda} \lambda_1 + F^0_{\varepsilon \varepsilon \varepsilon}\right)
\]  

(12)

from this latter, two values \( \lambda^\pm_{3/2} \) are found, which finally destroy the coalescence. Hence, the series (9) reads:

\[
\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^{3/2} \lambda^\pm_{3/2} + \ldots
\]  

(13)

It describes two curves emanating from \((\lambda_0, 0)\), at which they have the same tangent.

3 **Stability analysis of Hamiltonian systems under small circulatory forces**

We exploit the results achieved in the previous Sections to investigate the stability of a discrete Hamiltonian system perturbed by circulatory forces. The equation of motion are:

\[
\mathbf{Mx} + \mathbf{K} (\alpha, \beta) \mathbf{x} = \mathbf{0}
\]  

(14)

\(^1\)This case also includes that of the defective eigenvalue, when first- and second-order perturbations are both singular (see Eq. (18)-(19) of [4]). For this reason, it is a rare singularity.
where $M = M^T$ is a definite positive mass matrix, $K \neq K^T$ the stiffness matrix, $x$ the Lagrangian displacement vector and the dot denotes differentiation with respect to the time $t$. The stiffness matrix is assumed to depend on two bifurcation parameters, $\alpha, \beta$, at least one of which is a multiplier of a nonconservative force; the other parameter can be either a multiplier of a conservative or nonconservative force, or even a geometric/elastic parameter, describing a family of mechanical systems. Both the parameters are assumed to be small, and the system to be Hamiltonian when they vanish; therefore $K_0 := K(0, 0) = K_0^T$ is a positive definite elastic stiffness matrix. As well known, the mass matrix can be made unitary by a suitable change of variable and the equations of motion rewritten as:

$$\ddot{y} + A(\alpha, \beta) y = 0$$

(15)

where $A := T^{-1}K T^{-T}$, $y := T^T x$, having used the Cholesky decomposition $M = T^T T$. The equation of motion (15) admits the exponential solution $y = w \exp(\sigma t)$, which leads to the standard eigenvalue problem:

$$(A(\alpha, \beta) - \lambda I) w = 0$$

(16)

where $\lambda := -\sigma^2$. Accordingly, the trivial equilibrium position is (marginally) stable if $\lambda$ is real and positive (entailing $\sigma$ is purely imaginary); negative real values or complex values of $\lambda$ denotes unstable equilibrium (since at least one of the two values of $\sigma$ has positive real part).

When $\alpha, \beta$ are zero, since $A_0 := A(0, 0) = A_0^T$ is positive definite, all its eigenvalues are positive. When the parameters are varied, stability can be lost either (a) via the passage through zero of $\lambda$ (divergence), (b) via the collision of two eigenvalues at $\lambda_0$ and the successive birth of an imaginary component. This latter mechanism is well-known to occur in flutter phenomenon, that, occurring at finite values of the nonconservative forces, entails that $\lambda_0$ is defective. Less studied, instead is the case in which $\lambda_0$ is a double semi-simple eigenvalue of a Hamiltonian system (that cannot admit defective eigenvalues). This occurrence, indeed, is likely to manifests itself in systems where geometric and elastic symmetries entail the existence of two identical eigenvectors in two different physical planes. An example is represented by a beam with squared or circular cross-section, equally constrained in two orthogonal principal planes. When a small nonconservative force is applied to it, $\lambda_0$ can split in two complex eigenvalues, thus entailing instability of the Hamiltonian system under vanishingly small forces! This is the essence of the Paradox of Nicolai [5], concerning a symmetric cantilever loaded by a small follower torque. In [5] an extensive perturbation analysis similar to that detailed here in Sec 2, were carried out, but, however, limited to the first-order. Now, we want to extend the analysis to an higher-order.

Let us assume that $\lambda_0 > 0$ is a double semi-simple eigenvalue for the Hamiltonian system of matrix $A_0$. In the $(\alpha, \beta)$ parameter-plane we chose an exploring curve outgoing from the origin, of parametric equations:

$$\alpha = \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \ldots, \quad \beta = \varepsilon \beta_1 + \varepsilon^2 \beta_2 + \ldots$$

(17)

and we look for the special values of the coefficients, $\alpha_i, \beta_i$, which select the boundaries of the stable region, at which conditions of incipient instability occur. Along an exploring curve, $A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots$, where:

$$A_1 := A_0^\alpha \alpha_1 + A_0^\beta \beta_1, \quad A_2 := \frac{1}{2} \left( \alpha_1^2 A_0^\alpha + 2\alpha_1 \beta_1 A_0^\alpha \beta_1 + \beta_1^2 A_0^\beta + A_0^\alpha \alpha_2 + A_0^\beta \beta_2 \right)$$

(18)
3.1 First-order analysis

In the first-order analysis the exploring curve is a straight line, of director parameters \((\alpha_1, \beta_1)\). Along a generic direction, the double semi-simple eigenvalue \(\lambda_0\) split into the eigenvalues \(\lambda^\pm = \lambda_0 + \varepsilon \lambda_1^\pm + \ldots\), where \(\lambda_1^\pm\) are the solution to Eq. (6).

3.2 Second-order analysis

If one desires a more accurate representation of the stability region, an higher-order expansion for the eigenvalue must be used. However, we proved that the Taylor series fails, so that a Puiseux series must be used, along an exploring curve which is tangent to the singular direction. Along any curve tangent to it, the eigenvalue \(\lambda_0\) split into \(\lambda^\pm = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^{3/2} \lambda_3^{1/2} + \ldots\) (Eq. (13)), with \(\lambda_1\) real and \(\lambda_3^{1/2}\) given by Eq. (12). Since the Eq. (12) is a second order polynomial equation in the unknown \(\lambda_3^{1/2}\) these cases can occur: (a) if the radicand is positive, then \(\lambda^\pm\) are stable; (b) if it is negative one of them is unstable. Therefore, the condition of coalescence of the two roots again determines the incipient instability. Since this is a linear and inhomogeneous equation in \((\alpha_2, \beta_2)\) (being \((\alpha_1, \beta_1)\) already been determined), it is of type \(k_1 \alpha_2 + k_2 \beta_2 + k_3 = 0\), with \(k_i\) constants.

By letting, for example, \(\beta_2 = 0\), then \(\alpha_2 = -k_3/k_1\) is derived (if \(k_1 \neq 0\)). By substituting these results in the parametric equations (17), a second-order approximation of the bifurcation locus is finally obtained.

4 A sample system

As a sample example, we consider a four-d.o.f. system made of two planar double pendula, parallel to the \((y, z)\)-plane (Fig 2). The four rods, of equal length \(l\), are rigid and massless; lumped masses \(m\) are weightless; the elastic springs are linear, of stiffness \(\kappa\), except for one of them, of stiffness \(\kappa(1 + \alpha)\), so that the nondimensional parameter \(\alpha\) is a measure of the structural asymmetry. All the hinges are cylindrical, of \(x\)-axis. The two pendula are linked at their free ends by a telescopic tube, initially parallel to the \(x\)-axis, at which a follower force of intensity \(2F\) is applied, directed along the current position of the tube. Each pendulum is solicited in its own plane by the configuration-dependent \(y\)-component \(F_y\) of the force \(F\); this is the non-conservative (circulatory) part of the forces acting on the system.

The linearized equations of motion, in nondimensional form, reads:

\[
M\ddot{x} + (K_0 + \alpha K_\alpha + \beta K_\beta)x = 0
\]

where: \(M\) is the mass matrix; \(K_0\) is the stiffness matrix of the symmetric \((\alpha = 0)\) system; \(\alpha K_\alpha\) is the contribution to the stiffness due to the structural asymmetry; finally \(\beta K_\beta\) is the circulatory matrix. All matrices, except for \(K_\beta\) are symmetric; they are defined as follows:

\[
M := \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad K_0 := \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\]

\[
K_\alpha := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_\beta := \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}
\]

and \(x := (\theta_1, \theta_2, \theta_3, \theta_4)^T\) is the Lagrangian parameter vector, collecting the rotations \(\theta_i\) of the four
roads. Moreover, \( \beta := \frac{F}{m \omega^2} \) is a nondimensional load parameter, and the dot denotes differentiation with respect to the nondimensional time \( t := \omega \tilde{t} \), where \( \tilde{t} \) is the dimensional time and \( \omega := \sqrt{\frac{1}{l \sqrt{\kappa m}}} \).

By using the lower triangular matrix:

\[
\mathbf{T} = \begin{bmatrix} \hat{T} & 0 \\ 0 & \hat{T} \end{bmatrix}, \quad \hat{T} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}
\]

in the Cholesky transformation, the Eqs (19) are transformed into the Eqs (15).

4.1 First-order analysis

In the \((\alpha, \beta)\)-plane, we take straight exploring lines from the origin, of parametric equations:

\[
\alpha = \varepsilon \alpha_1, \quad \beta = \varepsilon \beta_1
\]

where \( \varepsilon \) is the parameter, and \( \alpha_1 \) or \( \beta_1 \) can be taken arbitrarily; consequently, \( \mathbf{A} = \mathbf{A}_0 + \varepsilon \mathbf{A}_1 \). The matrix \( \mathbf{A}_0 \) admits two double semi-simple eigenvalues; we consider the lowest eigenvalue (which is found to be significant for stability), i.e. \( \lambda_0 = 3 - \sqrt{2} \). The singularity condition reads:

\[
(F^0_{\alpha \epsilon})^2 - F^0_{\epsilon \epsilon} F^0_{\lambda \lambda} = \frac{\alpha_1}{8} \left( \left( 3 - 2\sqrt{2} \right) \alpha_1 - 4\beta_1 \right) = 0
\]

which is satisfied on two directions of the parameter plane:

\[
\mathbf{n}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}^{(2)} = \begin{pmatrix} 4 (3 + 2\sqrt{2}) \\ 1 \end{pmatrix}
\]

in which \( \beta_1 = 1 \) has been taken.

4.2 Second-order analysis

To perform a second-order analysis, we take an exploring curve which is no more a straight line, but which is tangent to one of the singular directions already determined:
\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \varepsilon \mathbf{n}^{(i)} + \varepsilon^2 \begin{pmatrix}
\alpha_2 \\
\beta_2
\end{pmatrix} \quad i = 1, 2
\] (25)

where, again \(\alpha_2\) or \(\beta_2\) can be taken arbitrarily.

We start by considering the direction \(\mathbf{n}_1^{(1)}\) in Eq (24). Here \(\alpha_1 = 0, \beta_1 = 1\) and therefore \(\lambda_1 = 0\).

The Eq (12) furnishes:

\[
\lambda_{3/2} = \pm i \frac{\sqrt{\alpha_2}}{2\sqrt{2}}
\] (26)

The condition of coalescence of the two roots supplies \(\alpha_2 = 0\); since \(\beta_2\) is arbitrary, we take \(\beta_2 = 0\).

Second, we consider the direction \(\mathbf{n}_2^{(2)}\) in Eq (24), on which \(\lambda_1 = \frac{2 + \sqrt{2}}{2}\).

Therefore, from Eq (12), it follows:

\[
\lambda_{3/2} = \pm i \frac{\sqrt{2}}{4} \sqrt{48 + 32\sqrt{2} + \left(12 + 8\sqrt{2}\right)\beta_2 - \alpha_2}
\] (27)

The coalescence of the two roots leads to:

\[
\alpha_2 = 48 + 32\sqrt{2}, \quad \beta_2 = 0
\] (28)

where we chose an arbitrary value for \(\beta_2\).

By collecting first- and second-order results, the stability boundaries, in parametric form, read:

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \varepsilon \begin{pmatrix}
0 \\
1
\end{pmatrix} + \varepsilon^2 \begin{pmatrix}
0 \\
0
\end{pmatrix}
\] (29)

and:

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \varepsilon \begin{pmatrix}
4 \left(3 + 2\sqrt{2}\right) \\
1
\end{pmatrix} + \varepsilon^2 \begin{pmatrix}
48 + 32\sqrt{2} \\
0
\end{pmatrix}
\] (30)

Equivalently, in Cartesian form:

\[
\alpha = 0 \quad \alpha = 4 \left(3 + 2\sqrt{2}\right) \beta + \left(48 + 32\sqrt{2}\right) \beta^2
\] (31)

Equations (31) have been plotted in Fig 3. Lines labelled by I denote first-order approximation; lines labelled by II, second-order approximations. The grey zone is the exact stability region, evaluated by numerically solving the eigenvalue problem. It is seen that the perturbation methods gives an excellent approximation in a large range of the asymmetry-parameter \(\alpha\) and in a smaller range of the load-parameter \(\beta\). It is also seen that a however small perturbation of the symmetric unloaded system can produce instability. This is the essence of the Nicolai paradox.

5 Conclusions

The eigenvalue (and eigenvector) sensitivities of linear systems depending on parameters have been evaluated via a perturbation approach. The important occurrence of double eigenvalues, naturally encountered in stability analysis of quasi-Hamiltonian systems, has been addressed. Two main cases must be distinguished, namely (a) not-semi-simple (or defective) eigenvalue, for which just one proper eigenvector exists, and (b) semi-simple (or non-defective) eigenvalue, for which two independent eigenvalues does exist. The singular character of the perturbation in the semi-simple case
Figure 3: Stability domain for the sample mechanical system; I: first-order and II: second-order perturbation solution; grey region: exact stability domain.

has also been investigated in the context of stability of nearly-Hamiltonian systems perturbed by circulatory forces. A mechanical system possessing four degrees of freedom, loaded by a follower force has been studied, and its stability domain determined. It has been shown that second-order perturbation analysis give results in excellent agreement with exact, numerical, solutions.

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References


